Chapter 1

Rooted forests and trees

The notion of a rooted forest should be familiar to the reader. For completeness, we will give formal definitions.

Let N be a finite set. A *rooted tree on* N is defined by the pair (N, p) where p is a function $p : N \longrightarrow N \cup \{\perp\}$ such that

- there is no positive integer k such that $p^k(x) = x$ for some element $x \in N$, and
- there is exactly one element $x \in N$ such that $p(x) = \perp$.

This element is called the *root*. The elements of N are called the *nodes* of the tree. They are also called the *vertices* of the tree.

For each nonroot node x, $p(x)$ is called the *parent* of x in the tree, and the ordered pair $xp(x)$ is called the *parent arc* of x in the tree. An *arc* of the tree is the parent arc of some nonroot node. If the parent of x is y then x is a *child* of y and xy is the *child arc* of y.

A rooted tree has *arity* k if every node has at most k children. A *binary* tree is a tree that has arity 2.

We say two nodes are *adjacent* if one is the parent of the other. We say the arc $xp(x)$ is *incident* to the nodes x and $p(x)$.

The *ancestors* of x are defined inductively: x is its own ancestor, and (if x is not the root) the ancestor's of x's parent are also ancestors of x. If x is the ancestor of y then y is a *descendant* of x. We say y is a *proper* ancestor of x (and x is a *proper* descendant of y) if y is an ancestor of x and $y \neq x$. The *depth* of a node is the number of proper ancestors it has.

We say an arc $xp(x)$ is an *ancestor arc* of y if x is an ancestor of y. We say $xp(x)$ is a *descendant arc* of y if $p(x)$ is a descendant of y.

A *subtree* of (N, p) is a tree (N', p') such that N' is a subset of N. A *rooted forest* is a collection of disjoint rooted trees. That is, (N, p) is a forest if there are trees $(N_1, p_1), \ldots, (N_k, p_k)$ such that

$$N = N_1 \dot{\cup} N_2 \dot{\cup} \cdots \dot{\cup} N_k$$

and p_i is the restriction of p to N_i.

Deletion of an arc $xp(x)$ from a rooted forest (N, p) is an operation that yields the forest (N, p') where

$$p'(x) = \begin{cases} \perp & \text{if } x = \hat{x} \\ p(x) & \text{otherwise} \end{cases}$$

If T is a rooted forest and e is an arc of T then we use $T - \{e\}$ to denote the result of deleting e.

Deletion of a node $\hat{x}$ from a rooted forest (N, p) is an operation that yields the forest $(N - \{\hat{x}\}, p')$ where

$$p'(x) = \begin{cases} \perp & \text{if } p(x) = \hat{x} \\ p(x) & \text{otherwise} \end{cases}$$

If T is a rooted forest and $\hat{x}$ is a node of T then we use $T - \{x\}$ to denote the result of deleting x.

More generally, if S is a set of nodes or a set of arcs, $T - S$ denotes the forest obtained by deleting every element of S.

For a tree T and a node x of T, the *subtree rooted at* x is the tree obtained from T by deleting every node that is not a descendant of x.

For a forest T and a node x of T, the *root-to-x path* is the sequence $x_0 x_1 \ldots x_k$ where x_0 is the root of T, x_k is x, and x_i is the parent of x_{i+1} for $i = 0, \ldots, k-1$. We denote this path by $T[x]$.

Ancestorhood defines a partial order among nodes of a forest. Given a set S of nodes of a forest, a *rootmost* node of S in the forest is a node v such that no proper ancestor of v is in S. A *leafmost* node of S is a node v such that no proper descendant of v is in S.

Given two nodes u and v of a forest, we say u is *leafward* of v and v is *rootward* of u if u is a descendant of v. A sequence $v_1, \ldots, v_k$ of nodes of the forest is a *leafward* path if v_i's parent is v_{i+1} for $i = 1, \ldots, k-1$.

1.1 Rootward computations

Suppose T is a rooted tree and $w(\cdot)$ is an assignment of weights to the nodes. There is a simple, linear-time algorithm to compute, for each node u, the total weight of all descendants of u:

```
def totalWeight(u):
    return w(u) + ∑{totalWeight(v) : v a child of u}
```

This algorithmic schema, though simple, comes up again and again: in finding separators for trees (in the next section), in algorithms that exploit interdigitating trees in planar graphs (Section 4.5, in processing a breadth-first-search tree (Section 5.4), in dynamic-programming algorithms on trees (Section 14.1) and on graphs of bounded carvingwidth (Section 14.3.1) and bounded branchwidth (Section 14.5.1).

1.2 Separators for rooted trees

A *separator* for a tree is a vertex or edge whose deletion results in trees that are "small" in comparison to the original graph.

Lemma 1.2.1 (Leafmost Heavy Vertex)**.** *Let T be a rooted tree. Let $\hat{w}(\cdot)$ be an assignment of weights to vertices such that the weight of each vertex is at least the sum of the weights of its children. Let W be the weight of the root, and let α be a positive number less than 1. Then there is a linear-time algorithm to find a vertex v_0 such that $\hat{w}(v_0) > \alpha W$ and every child v of v_0 satisfies $\hat{w}(v) \leq \alpha W$.*

Proof. Call the procedure below on the root of T.

```
define f(v):
1    if some child u of v has ŵ(u) > αW,
2         return f(u)
3         else return v
```

By induction on the number of invocations, for every call $f(v)$, we have $\hat{w}(v) > \alpha W$. If v is a leaf then the condition in Line 1 is not satisfied, so the procedure terminates. Let v_0 be the vertex returned by the procedure. Since the condition in Line 1 did not hold for v_0, every child v of v_0 satisfies $\hat{w}(v) \leq \alpha W$. □

1.2.1 Vertex separator

Lemma 1.2.2 (Tree Vertex Separator)**.** *Let T be a rooted tree, and let $w(\cdot)$ be an assignment of weights to vertices. Let W be the sum of weights. There is a linear-time algorithm to find a vertex v_0 such that every component in $T - \{v_0\}$ has total weight at most $W/2$.*

Proof. For each vertex u, define $\hat{w}(u) = \sum\{w(v) \ : \ v \text{ a descendant of } u\}$. Then $\hat{w}(\text{root}) = W$. The values $\hat{w}(\cdot)$ can be computed using a rootward computation as in Section 1.1. Let v_0 be the vertex of the Leafmost-Heavy-Vertex Lemma with $\alpha = 1/2$. Let $v_1, \ldots, v_p$ be the children of v_0. For each child v_i, the subtree rooted at v_i has weight at most $W/2$. Each such subtree is a tree of $T - \{v_0\}$. The remaining tree is $T - \{v \ : \ v \text{ is a descendant of } v_0\}$. Since the sum $\sum_{v \text{ is a descendant of } v_0} w(v) = \hat{w}(v_0)$ exceeds $W/2$, the weight of the remaining tree is less than $W/2$. □

1.3 Edge separators

For some separators, we need to impose a condition on the weight assignment. We say a weight assignment is *α-proper* if no element is assigned more than an α fraction of the total weight.

Lemma 1.3.1 (Tree Edge Separator of Edge-Weight)**.** *Let T be a tree of degree at most three, and let $w(\cdot)$ be a $\frac{1}{3}$-proper assignment of weights to edges. There is a linear-time algorithm to find an edge $\hat{e}$ such that every component in $T - \{\hat{e}\}$ has at most two-thirds of the weight.*

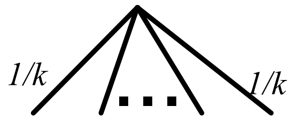
Proof. Assume for notational simplicity that the total weight is 1. Choose a vertex of degree one as root. For each nonroot vertex v, define

$$\hat{w}(v) = \sum \{w(e) \ : \ e \text{ a descendant edge of } v\} \cup \{\text{parent edge of } v\}$$

Define $\hat{w}(root) = 1$. Let v_0 be the vertex of the Leafmost-Heavy-Vertex Lemma with $\alpha = 1/3$. Let e_0 be the parent edge of v. Then $T - \{e_0\}$ consists of two trees. One tree consists of all descendants of v_0, and the other consists of all nondescendants.

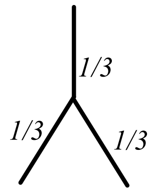
The weight of all edges among the nondescendants is $1 - \hat{w}(v_0)$, which is less than $1 - 1/3$ since $\hat{w}(v_0) > 1/3$. Let $v_1, \ldots, v_p$ be the children of v_0. (Note that $1 \leq p \leq 2$.) The weight of all edges among the descendants is $\sum_{i=1}^{p} \hat{w}(v_i)$. Since $\hat{w}(v_i) \leq 1/3$ for $i = 1, \ldots, p$ and $p \leq 2$, we infer $\sum_i \hat{w}(v_i) \leq 2/3$. □

The following example shows that the restriction on the arity of the trees in Lemma 1.3.1 cannot be discarded:



If the number of children is k then removal of any edge leaves weight $(k-1)/k$.

The following example shows that, for trees of degree at most three, the factor two-thirds in Lemma 1.3.1 cannot be improved upon.



Lemma 1.3.2 (Tree Edge Separator of Vertex Weight)**.** *Let T be a tree of degree at most three, and let $w(\cdot)$ be a $\frac{3}{4}$-proper assignment of weights to vertices such that each nonleaf vertex is assigned at most one-fourth of the weight. There is a linear-time algorithm to find an edge $\hat{e}$ such that every component $T - \{\hat{e}\}$ has at most three-fourths of the weight.*

Proof. Assume the total weight is 1. Root T at a leaf. For each vertex v, define

$$\hat{w}(v) = \sum \{w(v') \ : \ v' \text{ a descendant of } v\}$$

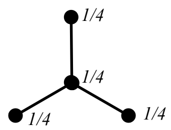
Let v be the vertex of the Leafmost-Heavy-Vertex Lemma with $\alpha = 3/4$. Let $v_1, \ldots, v_p$ be the children of v. Note that $0 \leq p \leq 2$. Since $w(v) \leq \frac{3}{4}$ but $\hat{w}(v) > \frac{3}{4}$, we must have $p > 0$, so $w(v) \leq \frac{1}{4}$.

For $1 \leq i \leq p$, let W_i be the weight of descendants of v_i. Let $\hat{\imath} = \text{maxarg}_{1 \leq i \leq p} W_i$. By choice of v, $W_{\hat{\imath}} \leq \frac{3}{4}$. By choice of $\hat{\imath}$,

$$W_{\hat{\imath}} \geq \frac{1}{2} \sum_{i=1}^{p} W_i > \frac{1}{2}(\frac{3}{4} - w(v_0)) \geq \frac{1}{2}(\frac{3}{4} - \frac{1}{4}) = W/4$$

This shows that choosing $\hat{e}$ to be the edge $v_{\hat{\imath}} v$ satisfies the balance condition. □

The following example shows that the factor three-fourths in Lemma 1.3.2 cannot be improved upon.



By changing our goal slightly, we can get a better-balanced separator.

Lemma 1.3.3 (Tree edge separator of Vertex/Edge Weight). *Let T be a binary tree, and let $w(\cdot)$ be a $\frac{1}{3}$-proper assignment of weight to the vertices and edges such that degree-three vertices are assigned zero weight. There is a linear-time algorithm to find an edge e such that every component of $T - e$ has at most two-thirds of the weight.*

Problem 1.3.4. *Prove Lemma 1.3.3.*

1.4 Recursive tree decomposition

Problem 1.4.1. *A recursive edge-separator decomposition for an unrooted tree T is a rooted tree D such that*

- *the root r of D is labeled with an edge e of T;*
- *for each connected component K of $T - e$ (there are at most two), r has a child in D that is the root of a recursive edge-separator decomposition of K.*

Show that there is an $O(n \log n)$ algorithm that, given a tree T of maximum degree three and n nodes, returns a recursive edge-separator decomposition of depth $O(\log n)$

1.5 Data structure for sequences and rooted trees

In the Appendix, we describe data structures for representing sequences and rooted trees.

Problem 1.5.1. *Show that the data structure for representing trees can be used to quickly find edge-separators in binary trees. Use this idea to give a fast algorithm that, given a tree of maximum degree three, returns a recursive edge-separator decomposition of depth $O(\log n)$.* Note: *A running time of $O(n)$ can be achieved.*