Chapter 2

Basic graph definitions

To quote Berge,

It would be convenient to say that there are two theories and two kinds of graphs: directed and undirected. This is not true. All graphs are directed, but sometimes the direction need not be specified.

That is, for specific graph problems it is convenient to ignore the distinction between endpoints.

We define one combinatorial structure, a graph.\(^1\) There are three ways to interpret this combinatorial structure, as an undirected graph, as a directed graph, and as a bidirected graph. Each kind of graph has its uses, and it is convenient to be able to view the underlying graph from these different perspectives.

In the traditional definition of graphs, vertices are in a sense primary, and edges are defined in terms of the vertices. We used this approach in defining rooted trees in Chapter 1. In defined graphs, we choose to make edges primary, and we will define vertices in terms of edges.

There are three reasons for choosing the edge-centric view:

- Self-loops and multiple edges, which occur often, are more simply handled by an edge-centric view.
- Contraction, a graph operation we discuss later, transforms a graph in a way that changes the identity of vertices but not of edges. The edge-centric view is more natural in this context, and simplifies the tracking of an edge as the graph undergoes contractions.
- The dual of an embedded graph is usefully viewed as a graph with the same edges, but where those edges form a different topology.

There is one seeming disadvantage: our definition of graphs does not permit the existence of isolated vertices, vertices with no incident edges. This disadvantage

\(^1\)Our definition allows for self-loops and multiple edges, a structure traditionally called a multigraph.
is mitigated by another odd aspect of our approach: a subgraph of a graph is not in itself an independent graph but depends parasitically on the original graph.

2.1 Edge-centric definition of graphs

For any finite set $E$, a graph on $E$ is a pair $G = (V, E)$ where $V$ is a partition of the set $E \times \{1, -1\}$, called the dart set of $G$. That is, $V$ is a collection of disjoint, nonempty, mutually exhaustive subsets of $E \times \{1, -1\}$. Each subset is a vertex of $G$. (The word node is synonymous with vertex). For any $e \in E$, the darts of $e$ are the pairs $(e, +1)$ and $(e, -1)$, of which the primary dart of $e$ is $(e, +1)$. For brevity, we can write $(e, +1)$ as $e^+$ and $(e, -1)$ as $e^-$. 

Problem 2.1. Write pseudocode to implement BFS given a graph represented as a partition of its dart set

\[ \begin{align*}
\text{rev} & \quad \text{Define the bijection rev on darts by } \text{rev}((e, \sigma)) = (e, -\sigma). \text{ For a dart } d, \\
\text{rev}(d) & \quad \text{is called the reverse of } d, \text{ and is sometimes written as } d^R.
\end{align*} \]

\[ \begin{align*}
\text{endpoints, head and tail, self-loops, parallel edges} & \quad \text{The head of a dart } (e, \sigma) \text{ is the block } v \in V \text{ such that } v \text{ contains } (e, \sigma). \text{ The tail of } (e, \sigma) \text{ is the head of } (e, -\sigma).
\end{align*} \]

Each element $e \in E$ has two endpoints, namely the head and tail of $(e, 1)$. If the endpoints are the same vertex, we call $e$ a self-loop. In Figure 2.1, $i$ is a self-loop. If two elements have the same endpoints, we say they are parallel, for example, $b$ and $j$ are parallel in Figure 2.1.

\[ \begin{align*}
\text{Edges and arcs} & \quad \text{We can interpret an element } e \in E \text{ as a directed arc, in which case we distinguish between its head and tail, which are, respectively, the}
\end{align*} \]
head and tail of the primary dart \((e,+1)\). If we interpret \(e\) as an undirected edge, we do not distinguish between its endpoints. Thus use of the word edge or arc indicates whether we intend to interpret the element as undirected or directed. The edge or arc of a dart \((e,\sigma)\) is defined to be \(e\).

**Parallel arcs/edges and self-loops**  If two arcs have the same tail and the same head, we say they are parallel arcs. If two edges have the same pair of endpoints, we say they are parallel edges. If the endpoints of an edge/arc are the same, we say it is a self-loop. Our definition of graph permits parallel edges and self-loops.

**Incidence, degree**  We say an edge/arc/dart is incident to a vertex \(v\) if \(v\) is one of the endpoints. The degree of a vertex \(v\) (written degree\((v)\)) is the number of occurrences of \(v\) as an endpoint of elements of \(E\) (counting multiplicity\(^2\)). The outdegree of \(v\) (written outdegree\((v)\)) is the number of arcs having \(v\) as a tail, and the indegree (written indegree\((v)\)) is the number of arcs having \(v\) as a head.

**Endpoint notation**  We sometimes write an arc as \(uv\) to indicate that its tail is \(u\) and its head is \(v\), and we sometimes write an edge the same way to indicate that its endpoints are \(u\) and \(v\). This notation has the potential to be ambiguous because of the possibility of parallel edges.

**\(V(G)\) and \(E(G)\)**  For a graph \(G = (V, E)\), we use \(V(G)\) and \(E(G)\) to denote \(V\) and \(E\), respectively, and we use \(n(G)\) and \(m(G)\) to denote \(|V(G)|\) and \(|E(G)|\). We use \(D(G)\) to denote the set of darts of \(G\). We may leave the graph \(G\) unspecified if doing so introduces no ambiguity.

![Figure 2.2](image_url)  Two graphs corresponding to the edges \(a,\ldots,e\).

### 2.2  Walks, paths, and cycles

**Walks**  As illustrated in Figure 2.1, a non-empty sequence

\[
d_1 \ldots d_k
\]

\(^2\)That is, a self-loop contributes two to the degree of a vertex.
of darts is a walk if the head of $d_i$ is the tail of $d_{i+1}$ for every $1 \leq i \leq k$. To be more specific, it is a $x$-to-$y$ walk if $x$ is $d_1$ or the tail of $d_1$ and $y$ is $d_k$ or the head of $d_k$. We define $d_1$ to be the successor in $W$ of $d_i$ to be $d_{i+1}$ and we define predecessor of $d_{i+1}$ to be $d_i$. We may designate a walk to be a closed walk if the tail of $d_1$ is the head of $d_k$, in which case we define the successor of $d_k$ to be $d_1$ and the predecessor of $d_1$ to be $d_k$. We also refer to a closed walk as a tour.

Paths and cycles A walk is called a path of darts if the darts are distinct, a cycle of darts if in addition it is a closed walk. A path/cycle of darts is called a path/cycle of arcs if each dart is of the form $(e, +1)$. It is called a path/cycle of edges if no edge is represented twice.

Simple paths and cycles, internal vertices A cycle is simple if every vertex occurs at most once as the head of some $d_i$. A path is simple if it is not a cycle and every vertex occurs at most once as the head of some $d_i$. A vertex is said to belong to the path or cycle if the vertex is an endpoint of some $d_i$. The internal vertices of a path $d_1 \ldots d_k$ are the heads of $d_1, \ldots, d_{k-1}$. Two paths/cycles are dart-disjoint if they share no darts, and are vertex-disjoint if they share no vertices. Two paths are internally vertex-disjoint if they share no internal vertices.

Walks, paths, and cycles of arcs/edges A sequence $e_1, \ldots, e_k$ of elements of $E$ is a directed walk (or diwalk) if the sequence of corresponding darts $(e_1, 1), \ldots, (e_k, 1)$ is a walk. It is a directed path (or dipath) if, in addition, $e_1, \ldots, e_k$ are distinct. It is an undirected walk if there exist $i_1, \ldots, i_k \in \{1, -1\}$ such that the sequence of darts $(e_1, i_1), \ldots, (e_k, i_k)$ is a walk. It is an undirected path if in addition $e_1, \ldots, e_k$ are distinct. The other definitions given for sequences of darts apply straightforwardly to paths consisting of elements of $E$.

Empty walks and paths In the above, we neglected to account for the possibility of an empty walk or path. Empty walks and paths are defined by a vertex in the graph; they contain no darts. We do not allow for the existence of empty cycles.

Lemma 2.2.1. A $u$-to-$v$ walk of darts contains a $u$-to-$v$ path of darts as a subsequence.

2.2.1 Connectedness

Given a graph $G = (V, E)$, for a vertex or dart $x$ and a vertex or dart $y$, we say $x$ and $y$ are connected in $G$ if there is a $v_1$-to-$v_2$ path of darts in $G$. Similarly, edges $e_1$ and $e_2$ are connected in $G$ if there is a path of darts that starts with a dart of $e_1$ and ends with a dart of $e_2$.

More generally, given a subset $E'$ of $E$, we say that $v_1, v_2$ are connected via $E'$ in $G$ if there is a $v_1$-to-$v_2$ path using only darts corresponding to edges of $E'$. 
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A subset of $V$ is connected in a graph if every two vertices in the subset are connected. Connectedness is an equivalence relation on the vertex set. A connected component is an equivalence class of this equivalence relation. Equivalently, a connected component is a maximal connected vertex subset. Let $\kappa(G)$ denote the number of connected components of $G$.

2.2.2 Subgraphs and edge subgraphs

We will use the term subgraph in two ways. According to the traditional definition, a subgraph of a graph $G = (V, E)$ is simply a graph $H = (V', E')$ such that $V' \subseteq V$ and $E' \subseteq E$. Because we often want to relate features of a subgraph to the graph from which it came, we will define an edge subgraph of $G$ as a pair $(G, E')$ where $E' \subseteq E(G)$.

If it is clear which graph $G$ is intended, we will sometimes use an edge-set $E'$ to refer to the corresponding edge subgraph $(G, E')$.

One significant distinction between a graph and an edge subgraph is this: according to our definition, a graph $G$ cannot contain a vertex with no incident edges, whereas an edge subgraph $(G, E')$ can contain a vertex $v$ (a vertex of $G$) none of whose incident edges belong to $E'$.

The usual definitions (walk, path, cycle, connectedness) extend to an edge subgraph by restricting the darts comprising these structures to those darts corresponding to edges in $E'$. For example, two vertices $x$ and $y$ of $G$ are connected in $(G, E')$ if there is an $x$-to-$y$ path of darts belonging to $E'$. As in graphs, a connected component of an edge subgraph of $G$ is a maximal connected subset of $V(G)$. We define $\kappa((G, E'))$ to be the number of connected components.
in this sense. For example, the edge subgraph on the bottom-left in Figure 2.3 has two connected components. (The graph on the bottom-right has only one.)

### 2.2.3 Deletion of edges and vertices

 Deleting a set $S$ of edges from $G$ is the operation on a graph that results in the subgraph or edge subgraph of $G$ consisting of the edges of $G$ not in $S$. We denote this subgraph or edge subgraph by $G - S$.

 The result of deleting a set $V'$ of vertices from $G$ is the graph (not the edge subgraph) obtained by deleting all the edges incident to the vertices in $V'$. This subgraph is denoted $G - V'$. Since isolated vertices (vertices with no incident edges) cannot exist according to our definition of graphs, deleted vertices cease to exist when deleted.

 Deletion of multiple edges and/or vertices results in a graph or edge-subgraph that is independent of the order in which the deletions occurred.

### 2.2.4 Contraction of edges

 For a graph $G = (V,E)$ and an edge $uv \in E$, the contraction of $e$ in $G$ is an operation that produces the graph $G' = (V',E')$, where

- $E' = E - \{uv\}$, and
- the part of $V$ containing $u$ and the part of $V$ containing $v$ are merged (and $uv$ is removed) to form a part $V'$.

\[\begin{align*}
&\text{contracting } a \\
&\text{contracting } e
\end{align*}\]

Figure 2.4: (a) A graph with edges $a, \ldots, e$. (b) The graph after the contraction of edge $a$.

Like deletions, the order of contractions of edges does not affect the result. For a set $S$ of edges, the graph obtained by contracting the edges of $S$ is denoted $G/S$. 
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2.2.5 Minors

A graph $H$ is said to be a minor of a graph $G$ if $H$ can be obtained from $G$ by edge contractions and edge deletions. The relation “is a minor of” is clearly reflexive, transitive, and antisymmetric.

Note that each vertex $v$ of $H$ corresponds to a set of vertices in $G$ (the set merged to form $v$).